A Note on Generalized Invariant Cones and the Kronecker Canonical Form

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ABSTRACT

We present conditions on the (generalized) spectrum of the pencil $A - \lambda B$ which are equivalent to the existence of an (A, B)-invariant proper convex cone K, i.e. $AK \subset BK$. This generalizes the notion of K-nonnegativity of A, i.e. $AK \subset K$.

1. INTRODUCTION

The classical Perron-Frobenius theory provides spectral conditions for a real $n \times n$ matrix A to be nonnegative elementwise, or equivalently for A to be $\mathbb{R}^n_+$-*nonnegative*, i.e. $A\mathbb{R}^n_+ \subset \mathbb{R}^n_+$. J. Vandergraft [5] and L. Elsner [1] independently extended this theory by deriving spectral conditions on a given real $n \times n$ matrix A which are necessary and sufficient for the

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existence of a proper cone $K \subset \mathbb{R}^n$ such that A is *K-nonnegative*, i.e. such that $AK \subset K$. (We say that K is *A-invariant*.) The conditions are given in terms of the spectrum of A.

In this paper we find necessary and sufficient conditions for the existence of a proper cone K such that the pair of real $m \times n$ matrices (A, B) is K-nonnegative, i.e. $AK \subset BK$, or K is *(A, B)-invariant*. The conditions are on the (generalized) spectrum of the pencil $A - \lambda B$ and are obtained using the (real) Kronecker canonical form. The conditions state that there are no infinite eigenvalues, all the left Kronecker indices are 0, and the finite eigenvalues satisfy the conditions given in [5] and [1].

2. PRELIMINARIES

We first present the required preliminaries on convex cones and the Kronecker form.

A set K in $\mathbb{R}^n$ is a *cone* if $\lambda K \subset K$ for all $\lambda \geqslant 0$. The cone K is called *proper* if it is closed, is convex, has nonempty interior, and is *pointed*, i.e. $K \cap \{-K\} = \{0\}$. The cone K is (A, B)-invariant if $AK \subset BK$, where A and B are real $m \times n$ matrices. We denote this by $(A, B) \overset{K}{\geqslant} 0$ and let Π^N denote the set of pairs of $m \times n$ matrices (A, B) for which there exists a proper cone K such that $(A, B) \overset{K}{\geqslant} 0$.

We now define the Kronecker canonical decomposition of the matrix pencil $A - \lambda B$; see e.g. [2]. Each such pencil is *strictly equivalent* to a quasidiagonal form, i.e.

$$S(A - \lambda B)T = \operatorname{diag}\left\{0, L_{\epsilon_1}, \dots, L_{\epsilon_r}, L^t_{\eta_1}, \dots, L^t_{\eta_s}, \lambda N - I, J - \lambda I\right\}, \quad (2.1)$$

with possible zero rows and columns at the top and at the left. Here S and T are nonsingular matrices, 0 represents a zero rectangular matrix, L_k is the $k \times (k+1)$ bidiagonal pencil

$$L_k = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad (2.2)$$

N is nilpotent, and both N and J are in Jordan canonical form.

The elementary divisors of $J - \lambda I$ are the *finite elementary divisors* of the pencil $A - \lambda B$ and yield the *finite eigenvalues*. The elementary divisors of $\lambda N - I$ are the *infinite elementary divisors* and yield the *infinite eigenvalues*. The index sets $\{\epsilon_1, \ldots, \epsilon_t\}$ and $\{\eta_1, \ldots, \eta_s\}$ are the nonzero *right and left Kronecker indices*, respectively. The 0 rectangular matrix contains the blocks corresponding to zero Kronecker indices, as do the possible zero rows and columns. A left (right) Kronecker index corresponds to the existence of a row (column) polynomial vector that zeros out the pencil identically.

For the real matrix pencil $A - \lambda B$, we can take J to be in real Jordan canonical form and S and T real. (See e.g. [3] for the definition of the real Jordan canonical form.) We then call (2.1) the *real Kronecker canonical form* in this case.

For a square matrix C, we let $\sigma(C)$ denote the spectrum of C, $\rho(C)$ denote the spectral radius of C, and $d(\lambda)$ denote the *degree* of an eigenvalue λ, i.e., $d(\lambda)$ is the size of the largest Jordan block corresponding to the eigenvalue λ.

3. GENERALIZED INVARIANT CONES

Vandergraft [5] and Elsner [1] have extended the classical Perron-Frobenius theory for nonnegative matrices and have provided necessary and sufficient conditions for a matrix A to be K-nonnegative. A basic tool used was the Jordan canonical form. We now use the (real) Kronecker canonical form to extend this theory for generalized invariance. We see that the conditions are unchanged for the finite eigenvalues of the pencil.

THEOREM 3.1. *Suppose that the pencil $A - \lambda B$ has the real Kronecker canonical form* (2.1). *Then the following are equivalent*:

(i) $(A, B) \in \Pi^N$.

(ii) *The left Kronecker indices are all* 0; *there are no infinite eigenvalues*; $\rho(J) \in \sigma(J)$, *and if* $\lambda \in \sigma(J)$ *is such that* $|\lambda| = \rho(J)$, *then* $d(\lambda) \leqslant d(\rho(J))$.

Proof. For nonsingular matrices S and T, we have that $AK \subset BK$ if and only if $\mathrm{SAT}(T^{-1}K) \subset SBT(T^{-1}K)$. Moreover, $T^{-1}K$ is a proper cone if K is. Therefore, without loss of generality, we can assume that the pencil $A - \lambda B$ is already in real Kronecker canonical form (2.1).

Since K is a proper cone and so has nonempty interior, we see that $AK \subset BK$ implies that the ranges satisfy

$$\mathscr{R}(A) \subset \mathscr{R}(B). \tag{3.1}$$

But for $k > 0$,

$$L_k^t = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & 1 \\ & \ddots & \\ & & 0 \end{bmatrix}, \tag{3.2}$$

i.e., if a left Kronecker index is > 0, then (3.1) is violated. Similarly, we see that we cannot have a block $\lambda N - I$, i.e., we cannot have any infinite eigenvalues.

Therefore, if $AK \subset BK$, then the pencil $A - \lambda B$ must be of the form

$$A - \lambda B = \operatorname{diag}\{0, L_{\epsilon_1}, \dots, L_{\epsilon_t}, J - \lambda I\}. \tag{3.3}$$

Now, we can adjoin the row $(0, \dots, 0, -\lambda)$ to the bidiagonal pencils L_k without affecting the existence of the (A, B)-invariant proper cone K. We can also add columns or rows of zeros and diagonal elements to ensure that the zero block is square and has the form $\operatorname{diag}\{-\lambda, \dots, -\lambda\}$. If columns were added, then we can take the cross product $K \times \mathbb{R}_+^n$ to replace K if needed. The new cone is proper if and only if K is. Note that the criterion for the existence of K is independent of any 0 eigenvalues. The pencil $A - \lambda B$ is now square and has the form $A - \lambda I$. The result now follows directly from the conditions given in [5] and [1]. ■

REMARKS. We can similarly extend the characterizations of the stronger properties of K-positivity, K-irreducibility, and K-strong nonnegativity which have been given in [5] and [1]. Also, we can apply the above techniques to the characterizations in [4] for the case when K is restricted to be an "ellipsoidal" cone, and to the characterizations in [1] for "exponential" nonnegativity.

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